Computing the Chern-Scwrartz-MacPherson Class and Euler Characteristic of Complete Simplical Toric Varieties

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In this note we present Algorithm 2.1, a combinatorial algorithm which computes the Chern-Schwartz-MacPherson (c_{SM}) class and/or the Euler characteristic of a complete simplicial toric variety X_Σ defined by a fan Σ (that is we allow X_Σ to have finite quotient singularities). The algorithm is based on a result of Barthel, Brasselet and Fieseler [1] which gives an expression for the c_{SM} class of a toric variety in terms of torus orbit closures. Note that we will only consider toric varieties X_Σ over the complex numbers $\mathbf{C}$.

We also note that the restriction to complete simplicial toric varieties is not required in the statement of the result of Barthel, Brasselet and Fieseler [1] on which our algorithm is based, indeed these restrictions are present on the algorithm only for the purpose of simplifying the construction of the Chow ring of the toric variety. If one was able to construct the Chow ring in a simple manner with the restrictions removed the algorithm could be applied unchanged in this more general setting.

The Macaulay2 [3] implementation of our algorithm for computing the c_{SM} class and Euler characteristic of a complete simplicial toric variety presented in this note can be found at `https://github.com/Martin-Helmer/char-class-calc`. This implementation is accessed via the “CharToric" package.

1 Setting and Notation

Let X_Σ be a n-dimensional complete and simplicial toric variety; then the intersection product can be defined on rational cycles (see §12.5 of [2]) so that, if we let $\mathbf{Q}$ denote the rational numbers and $\mathbf{Z}$ the integers, we have that the rational Chow ring of X_Σ is given by the graded ring

$$A^*(X_\Sigma)_{\mathbf{Q}} = A^*(X_\Sigma) \otimes_{\mathbf{Z}} \mathbf{Q} = \bigoplus_{j=0}^{n} A^j(X_\Sigma) \otimes_{\mathbf{Z}} \mathbf{Q}. \qquad (1)$$

For each cone σ in the fan Σ the orbit closure $V(\sigma)$ is a subvariety of codimension $\dim(\sigma)$. We will write $[V(\sigma)]$ for the rational equivalence class of $V(\sigma)$ in $A^{\dim(\sigma)}(X_\Sigma)$.

Proposition 1.1 (Lemma 12.5.1 of [2]) *The collections $[V(\sigma)] \in A_j(X_\Sigma)$ for $\sigma \in \Sigma$ having dimension $n-j$ generate $A_j(X_\Sigma)$, the Chow group of dimension j. Further the collection $[V(\sigma)]$ for all $\sigma \in \Sigma$ generates $A^*(X_\Sigma)$ as an abelian group.*

The following proposition gives us a simple method to compute the rational Chow ring of a complete, simplicial toric variety X_Σ.

Proposition 1.2 (Theorem 12.5.3 of Cox, Little, Schenck [2]) *Let N be an integer lattice with dual M. Let X_Σ be a complete and simplicial toric variety with generating rays $\Sigma(1) = \rho_1, \dots, \rho_r$ where $\rho_j = \langle v_j \rangle$ for $v_j \in N$. Then we have that*

$$\mathbf{Q}[x_1, \dots, x_r]/(\mathscr{I} + \mathscr{J}) \cong A^*(X_\Sigma)_{\mathbf{Q}}, \tag{2}$$

with the isomorphism map specified by $[x_i] \mapsto [V(\rho_i)]$. Here $\mathscr{I}$ denotes the Stanley-Reisner ideal of the fan Σ, that is the ideal in $\mathbf{Q}[x_1, \dots, x_r]$ specified by

$$\mathscr{I} = (x_{i_1} \cdots x_{i_s} \mid i_{i_j} \text{ distinct and } \rho_{i_1} + \cdots + \rho_{i_s} \text{ is not a cone of } \Sigma) \tag{3}$$

and J denotes the ideal of $\mathbf{Q}[x_1, \dots, x_r]$ generated by linear relations of the rays, that is $\mathscr{J}$ is generated by linear forms

$$\sum_{j=1}^{r} m(v_j) x_j \tag{4}$$

for m ranging over some basis of M.

2 Algorithm

In this section we present Algorithm 2.1 which computes the c_{SM} class and/or Euler characteristic of a complete simplicial toric variety defined by a fan Σ.

Proposition 2.1 (Main Theorem of Barthel, Brasselet and Fieseler [1]) *Let X_Σ be an n-dimensional complex toric variety specified by a fan Σ. We have that the Chern-Schwartz-MacPherson class of X_Σ can be written in terms of orbit closures as*

$$c_{SM}(X_\Sigma) = \sum_{\sigma \in \Sigma} [V(\sigma)] \quad \in A^*(X_\Sigma)_{\mathbf{Q}} \tag{5}$$

where $V(\sigma)$ is the closure of the torus orbit corresponding to σ.

Lemma 2.2 is a modified version of Proposition 11.1.8. of Cox, Little, Schenck [2], it will allow us to compute the multiplicity of a simplicial cone. We have slightly altered the statement of the result to explicitly show how we will compute these multiplicities in practice.

Lemma 2.2 (Modified version of Proposition 11.1.8. of Cox, Little, Schenck [2]) *Let $N = \mathbf{Z}^n$ be an integer lattice. For a simplicial cone $\sigma = \rho_1 + \cdots + \rho_d \subset N$ let M_σ be the matrix with columns specified by the generating vectors of the rays $\rho_1, \ldots, \rho_d$ which define the cone σ; we have*

$$\mathrm{mult}(\sigma) = |\det(\mathrm{Herm}(M_\sigma))| \tag{6}$$

where $\mathrm{Herm}(M_\sigma)$ *denotes the Hermite normal form of matrix M_σ with all zero rows and/or zero columns removed. Further* $\mathrm{mult}(\sigma) = 1$ *if and only if U_σ is smooth.*

To compute the classes $[V(\sigma)]$ appearing in (5) we will employ the following proposition combined with Proposition 1.2.

Proposition 2.3 (Theorem 12.5.2. of Cox, Little, Schenck [2]) *Assume that X_Σ is complete and simplicial. If $\rho_1, \ldots, \rho_d \in \Sigma(1)$ are distinct and if $\sigma = \rho_1 + \cdots + \rho_d \in \Sigma$ then in $A^*(X_\Sigma)$ we have the following:*

$$[V(\sigma)] = \mathrm{mult}(\sigma)[V(\rho_1)] \cdot [V(\rho_2)] \cdots [V(\rho_d)]. \tag{7}$$

Here $\mathrm{mult}(\sigma)$ *will be calculated using Lemma 2.2.*

In Algorithm 2.1 we present an algorithm to compute $c_{SM}(X_\Sigma)$ for a complete, simplicial toric variety X_Σ defined by a fan Σ. Note that we represent $[V(\rho_j)]$ as x_j using the isomorphism in Proposition 1.2.

Algorithm 2.1 ***Input:*** *A complete, simplicial toric variety X_Σ defined by a fan Σ with $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$ and a boolean, Euler_only, indicating if only the Euler characteristic is desired. We assume* $\dim(X_\Sigma) \geq 1$.
Output: *$c_{SM}(X_\Sigma)$ in $A^*(X_\Sigma)_{\mathbf{Q}} \cong \mathbf{Q}[x_1, \ldots, x_r]/(\mathscr{I} + \mathscr{J})$ and/or the Euler characteristic $\chi(X_\Sigma)$, if Euler_only=true then only $\chi(X_\Sigma)$ will be computed.*

- *Compute the rational Chow ring $A^*(X_\Sigma)_{\mathbf{Q}} \cong \mathbf{Q}[x_1, \ldots, x_r]/(\mathscr{I} + \mathscr{J})$ using Proposition 1.2.*
- csm $= 0$.
- ***For** i **from*** $\dim(X_\Sigma)$ ***to*** 1***:***
  - orbits $=$ *all subsets of $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$ containing i elements.*
  - total $= 0$.
  - ***For** $\rho_{j_1}, \ldots, \rho_{j_s}$ **in*** orbits***:***
    - ◇ $\sigma = \rho_{j_1} + \cdots + \rho_{j_s}$.
    - ◇ *Find $w =$* $\mathrm{mult}(\sigma)$ *using Lemma 2.2.*

  - ◇ $[V(\sigma)] = \mathrm{mult}(\sigma)[V(\rho_{i_1})]\cdots[V(\rho_{i_s})] = w \cdot x_{i_1}\cdots x_{i_s}$.
  - ◇ $\mathrm{total} = \mathrm{total} + [V(\sigma)]$.
- ○ $\mathrm{csm} = \mathrm{csm} + \mathrm{total}$.
- ○ ***If*** $i == \dim(X_\Sigma)$***:***
  - ◇ *Set* $(c_{SM}(X_\Sigma))_0 = \mathrm{csm}$.
  - ◇ *Set* $\chi(X_\Sigma)$ *= sum of the coefficients of the monomials in* $(c_{SM}(X_\Sigma))_0$.
  - ◇ ***If Euler_only==true:***
    - ▷ ***Return*** $\chi(X_\Sigma)$.

- *Set* $c_{SM}(X_\Sigma) = \mathrm{csm}$.
- ***Return*** $c_{SM}(X_\Sigma)$ ***and/or*** $\chi(X_\Sigma)$.

We note that Algorithm 2.1 is strictly combinatorial; hence the runtime depends only on the combinatorics of the fan Σ defining the toric variety.

3 Performance

In this section we give the run times for Algorithm 2.1 applied to a variety of examples. Consider a complete simplicial toric variety X_Σ. We give two alternate implementations of Algorithm 2.1 to reflect what we can expect the timings to be in both the smooth cases and singular cases.

Specifically the running times in Table 1 for Algorithm 2.1 marked with a † check the input to see if the given fan Σ defines a smooth toric variety, if it does these implementations use the fact that $\mathrm{mult}(\sigma) = 1$ for all $\sigma \in \Sigma$ and hence do not compute the Hermite normal forms and their determinates in Lemma 2.2. However to show how the algorithm would perform on a singular input of a similar size and complexity we also give running times for an implementation which always computes the Hermite forms and their determinates in Lemma 2.2. In this way we see in a precise manner what the extra cost associated to computing the c_{SM} class and Euler characteristic of a singular toric variety would be in comparison to the cost of computing a smooth toric variety defined by a fan having similar combinatorial structure.

By default the implementation of Algorithm 2.1 in our “CharToric" package checks if the input defines a smooth toric variety, i.e. performs the procedure of the implementations marked with †.

We also remark that the extra cost in the singular case (or in the case where we don’t check the input) comes entirely from performing linear algebra with integer matrices. As such the running times in these cases could perhaps be somewhat reduced by using a specialized integer linear algebra package. To give a rough quantification of what performance improvement one might expect from this we performed some testing using LinBox [4] and PARI [6] via Sage [5] on linear systems of similar size and structure to those arising in the examples in Table 1. In this testing we found that the specialized algorithms seemed to be around two to three times faster than the linear algebra methods used by our implementation in the “CharToric" package, however this testing is by no means conclusive.

Input	Alg. 2.1 †	Alg. 2.1 (Euler only) †	Alg. 2.1	Alg. 2.1 (Euler only)	Chow Ring (Prop. 1.2)
$\mathbf{P}^6$	0.0s	0.0s	0.0s	0.0s	0.1 s
$\mathbf{P}^{16}$	5.3s	0.0s	85.4s	0.0s	0.7 s
$\mathbf{P}^5 \times \mathbf{P}^6$	0.3s	0.0s	3.7s	0.0s	1.2 s
$\mathbf{P}^5 \times \mathbf{P}^8$	1.1s	0.0s	16.8s	0.1s	2.1 s
$\mathbf{P}^8 \times \mathbf{P}^8$	12.0s	0.1s	168.5s	0.1s	4.5 s
$\mathbf{P}^5 \times \mathbf{P}^5 \times \mathbf{P}^5$	12.8s	0.2s	156.7s	0.6s	11.8 s
$\mathbf{P}^5 \times \mathbf{P}^5 \times \mathbf{P}^6$	28.4s	0.3s	387.1s	0.8s	17.0 s
Fano sixfold 123	0.3s	0.0s	1.0s	0.4s	1.1 s
Fano sixfold 1007	0.4s	0.1s	1.0s	0.1s	1.8 s

Table 1: Note that the table we present the time to compute the Chow ring seperately from the time reqired for the other computations, as such the total run time for each algorithm will be the time listed in its column plus the time to compute the Chow ring if the Chow ring is not already known. Computations were performed using Macaulay2 [3] on a computer with a 2.9GHz Intel Core i7-3520M CPU and 8 GB of RAM. The Fano sixfolds are those built by the smoothFanoToricVariety method in the “NormalToricVarieties" Macaulay2 [3] package. $\mathbf{P}^n$ denotes a projective space of dimension n

References

[1] Gottfried Barthel, J-P Brasselet, and K-H Fieseler. Classes de Chern de variétés toriques singulières. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, 315(2):187–192, 1992.

[2] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124. American Mathematical Soc., 2011.

[3] Daniel R. Grayson and Michael E. Stillman. *Macaulay2, a software system for research in algebraic geometry*, 2013.

[4] The LinBox Group. *LinBox – Exact Linear Algebra over the Integers and Finite Rings, Version 1.1.6*, 2008.

[5] W. A. Stein et al. *Sage Mathematics Software (Version 5.11)*. The Sage Development Team, 2013. `http://www.sagemath.org`.

[6] The PARI Group, Bordeaux. *PARI/GP version* `2.7.0`, 2014. available from `http://pari.math.u-bordeaux.fr/`.